.TH 155/GRACEY
EXAM 2/CHAPTER 9
100 POINTS POSSIBLE

NAME Key

SHOW ALL WORK, GIVE EXACT ANSWERS (UNLESS OTHERWISE INDICATED) AND SUPPORT ALL RESULTS TO EARN FULL CREDIT

1. (3 points) Determine the convergence or divergence of the **sequence** with the given nth term. If the sequence converges, find its limit.

$$a_n = \frac{(n-2)!}{n!}$$

$$\lim_{n\to\infty} \frac{(n-2)!}{n!} = \lim_{n\to\infty} \frac{\cancel{(n-2)!}}{n(n-1)\cancel{(n-2)!}} = \lim_{n\to\infty} \frac{1}{n(n-1)} = 0$$

$\boxed{a_n = \frac{(n-2)!}{n!} \text{ converges to } 0.}$

2. (6 points) Find the sum of the convergent telescoping series.

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \frac{1}{2}\sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$\boxed{\text{PFD}}$

$$\frac{1}{(n+1)(n-1)} = \frac{A}{n+1} + \frac{B}{n-1}$$

$$1 = An - A + Bn + B$$

$$1 = (A+B)n + (-A+B)$$

$$0 = A+B$$

$$1 = -A+B$$

$$1 = 2B$$

$$\frac{1}{2} = B \rightarrow -\frac{1}{2} = A$$

$$= \frac{1}{2}\left[\left(1-\cancel{\tfrac{1}{3}}\right) + \left(\frac{1}{2} - \cancel{\tfrac{1}{4}}\right) + \left(\cancel{\tfrac{1}{3}} - \frac{1}{5}\right) + \left(\cancel{\tfrac{1}{4}} - \frac{1}{6}\right) + \cdots\right]$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{(n+1)-1} - \frac{1}{(n+1)+1}\right)$$

$$= \frac{1}{2}\left(\frac{3}{2} + 0 - 0\right)$$

$$= \boxed{\frac{3}{4}}$$

3. (5 points) Consider the repeating decimal $0.\overline{63}$.

a. Write the repeating decimal as a geometric series.

$$0.\overline{63} = 0.63 + 0.0063 + 0.000063 + \cdots$$

$$= .63(1 + .01 + .0001 + \cdots)$$

$$= \boxed{\sum_{n=0}^{\infty}(0.63)(.01)^n}$$

b. Write its sum as the ratio of two integers.

$$S = \frac{0.63}{1-0.01}$$

$$S = \frac{.63}{.99}$$

$$\boxed{S = \frac{63}{99}}$$

1. (40 points, 10 points each) Determine the convergence or divergence of the series. Only work 4 out of the 5 items.

a. $\sum_{n=1}^{\infty} \frac{3^n}{2^n - 1}$

Step 1: Identify test and its conditions (if applicable)

DCT

Comparison series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$

$\left(\frac{3}{2}\right)^n < \frac{3^n}{2^n-1}$ for all n and both series have positive terms

Step 2: Run the test

$\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ is a divergent geometric series $\left[|r| = \frac{3}{2} \geq 1\right]$

Step 3: Conclusion

$\sum_{n=1}^{\infty} \frac{3^n}{2^n-1}$ diverges by the DCT

b. $\sum_{n=1}^{\infty} \frac{(n!)^2}{(3n)!}$

Step 1: Identify test and its conditions (if applicable)

Ratio test. This is a series with nonzero terms.

Step 2: Run the test

$$\lim_{n\to\infty} \left| \frac{[(n+1)!]^2}{[3(n+1)]!} \cdot \frac{(3n)!}{(n!)^2} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)!\,(n+1)!\,\cancel{(3n)!}}{(3n+3)(3n+2)(3n+1)\cancel{(3n)!}\,n!\,n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!} \cdot \cancel{(n+1)}\cancel{n!}}{3\cancel{(n+1)}(3n+2)(3n+1)\cancel{n!}\cancel{n!}} \right|$$

$= 0 < 1$

Step 3: Conclusion

$\sum_{n=1}^{\infty} \frac{(n!)^2}{(3n)!}$ converges by the ratio test.

c. $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{(2n-1)\pi}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Step 1: Identify test and its conditions (if applicable)

A.S.T.

Step 2: Run the test

$a_n = \frac{1}{n}$

1) $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

2) $a_{n+1} \le a_n$ for all n ✓

Step 3: Conclusion

$\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{(2n-1)\pi}{2}$ converges by the AST.

d. $\sum_{n=1}^{\infty} \left(\frac{2n}{n+1} \right)^n$

Step 1: Identify test and its conditions (if applicable)

root test

Step 2: Run the test

$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{2n}{n+1}\right)^n\right|} = \lim_{n\to\infty} \frac{2n}{n+1}$

$= 2 > 1$

Step 3: Conclusion

$\sum_{n=1}^{\infty} \left(\frac{2n}{n+1}\right)^n$ diverges by the root test.

e. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} (-1)^n n^{1/6}$

Step 1: Identify test and its conditions (if applicable)

AST

Step 2: Run the test

$a_n = n^{1/6}$

1) $\lim_{n\to\infty} n^{1/6} = \infty$

Step 3: Conclusion

$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{\sqrt[3]{n}}$ diverges by the nth term test for divergence

5. (6 points) Find the 4[th] Taylor polynomial for $f(x)=\ln x$, centered at 2.

$f(x)=\ln x$, $f(2)=\ln 2$

$f'(x)=\frac{1}{x}$, $f'(2)=\frac{1}{2}$

$f''(x)=-\frac{1}{x^2}$, $f''(2)=-\frac{1}{4}$

$f'''(x)=\frac{2}{x^3}$, $f'''(2)=\frac{1}{4}$

$f^{(4)}(x)=-\frac{6}{x^4}$, $f^{(4)}(2)=-\frac{3}{8}$

$P_4(x)=\ln 2+\frac{1}{2}(x-2)-\frac{1}{4}\frac{(x-2)^2}{2!}+\frac{1}{4}\frac{(x-2)^3}{3!}-\frac{3}{8}\frac{(x-2)^4}{4!}$

$\boxed{P_4(x)=\ln 2+\frac{1}{2}(x-2)-\frac{1}{8}(x-2)^2+\frac{1}{24}(x-2)^3-\frac{1}{64}(x-2)}$

6. (10 points) Find a power series for the function $f(x)=\dfrac{4}{3-x}$, centered at 1, and determine the interval of convergence.

$\frac{4}{3-1-(x-1)}=\frac{4/2}{\frac{2}{2}-\frac{(x-1)}{2}}$

$=\frac{2}{1-\frac{x-1}{2}}$

$\boxed{\sum_{n=0}^{\infty}2\left(\frac{x-1}{2}\right)^n,\ (-1,2)}$

I.O.C.

$\left|\frac{x-1}{2}\right|<1$

$-1<\frac{x-1}{2}<1$

$-2<x-1<1$

$-1<x<2$

7. (15 points) Use the power series $\frac{1}{1+x}=\sum_{n=0}^{\infty}(-1)^n x^n$ and $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$ to determine a power series, centered at 0, for the function $h(x)=\frac{-2}{x^2-1}=\frac{1}{1+x}+\frac{1}{1-x}$. Identify the interval of convergence.

$$\frac{1}{1+x}+\frac{1}{1-x}=\sum_{n=0}^{\infty}(-1)^n x^n+\sum_{n=0}^{\infty}x^n$$

$$=\sum_{n=0}^{\infty}\left[(-1)^n+1\right]x^n$$

$$=2x^0+0x^1+2x^2+0x^3+2x^4+\dots$$

$$=\boxed{2\sum_{n=0}^{\infty}x^{2n}\ ,\ (-1,1)}$$

8. (15 points) Use the power series $\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$ to find the Maclaurin series for the function $f(x)=\cos^2 x$.

$$\cos^2 x=\frac{1+\cos 2x}{2}$$

$$=\frac{1}{2}\left[1+\cos 2x\right]$$

$$=\frac{1}{2}\left[1+1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\frac{(2x)^6}{6!}+\dots\right]$$

$$=\boxed{\frac{1}{2}\left[1+\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right]}$$